MAT 217 Sample Problems

Note: These problems assume that you are familiar with the concepts of *vector space, linear independence, basis, dimension,* and *linear transformation*. Do not fear if these are new to you - they will be explained in detail in MAT 217 - but if you want to attempt these problems before taking the class you may need to read about these concepts either in an introductory linear algebra textbook such as Hoffman & Kunze or on Wikipedia.

1)

Let V and W be finite dimensional vector spaces and let $T : V \to W$ be a linear transformation. Define the *nullspace* of T, N(T), to be the set of all $\alpha \in V$ such that $T(\alpha) = 0$. Define the *range* of T, R(T), to be the set of all $\gamma \in W$ such that there exists a $\beta \in V$ with $T(\beta) = \gamma$. Prove that $N(T)$ and $R(T)$ are vector spaces (specifically, $N(T)$ is a subspace of V and $R(T)$ is a subspace of W). Then prove that $\dim V = \dim N(T) + \dim R(T)$.

2)

Let V be a vector space and T a linear transformation from V into V. Prove that the following two statements are equivalent: (i) The intersection of the nullspace of T and the range of T is $\{0\}$, and (ii) For $\alpha \in V$, if $T(T(\alpha)) = 0$ then $T(\alpha) = 0$.

3)

Let V and W be finite-dimensional vector spaces and let $T : V \to W$ be a linear transformation. Prove that: (a) if $\dim V = \dim W$ then T is one-to-one if and only if T is onto, (b) if $\dim V < \dim W$ then T cannot be onto, and (c) if $\dim V > \dim W$ then T cannot be one-to-one.

4)

Let V be a finite-dimensional vector space over the field F and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field and let $\beta_1, \beta_2, \ldots, \beta_n$ be any vectors in W. Prove that there is exactly one linear transformation $T : V \to W$ such that $T(\alpha_i) = \beta_i$ for all $1 \leq i \leq n$.

5)

Let V be the set of real numbers, and consider V as a vector space over the field of rational numbers. Prove that this vector space is not finite-dimensional (i.e. prove that it does not have a finite basis).